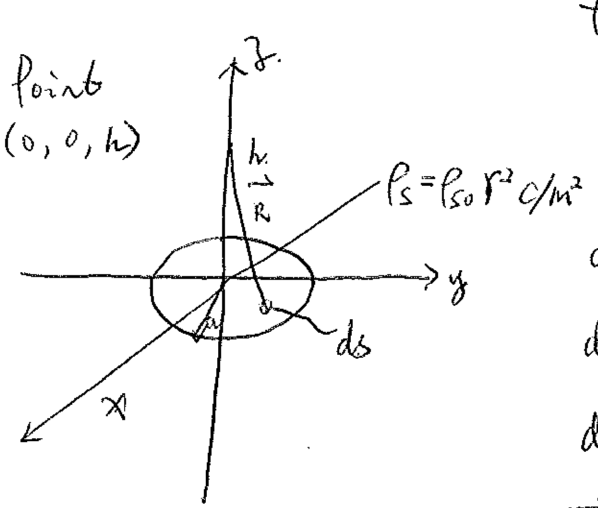
Problem 4.15 ECEN3613

free space $\Rightarrow \varepsilon = \epsilon_0$



$r\big|_0^a \quad , \phi\big|_0^{2\pi}$

$dS = r\,dr\,d\varphi \qquad \vec{R} = -r\tilde{a}_r + h\tilde{a}_z$

$$d\vec{E} = \frac{\rho_s\,dS}{4\pi\epsilon_0 R^2}\tilde{a}_R \qquad R = \sqrt{r^2+h^2}$$

$$d\vec{E} = \frac{\rho_s\,dS\,\vec{R}}{4\pi\epsilon_0 R^3}$$

$$d\vec{E}\big|_{z=h} = \frac{\rho_{s0} r^3(-r\hat{a}_r + h\hat{a}_z)\,dr\,d\varphi}{4\pi\epsilon_0(r^2+h^2)^{\frac{3}{2}}}$$

$$\vec{E}\big/_{z=h} = \frac{\rho_{s0}}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^a \frac{(-r^4\hat{a}_r + hr^3\hat{a}_z)}{(r^2+h^2)^{\frac{3}{2}}}\,dr\,d\varphi$$

By symmetry we can see that $\hat{a}_r$ component will be zero since a diametrically opposite ΔS at the same radius will have a radial component opposite that of the othe ΔS ! (The z-components will add) we can, of course, complete the integration of the $\hat{a}_r$ component showing the cancellation. $\Longrightarrow$

$$E_z = \frac{\rho_{s0}}{4\pi\epsilon_0}(2\pi)h\int_0^a \frac{r^3}{(r^2+h^2)^{3/2}}\,dr = \frac{\rho_{s0}h}{2\epsilon_0}\left[(r^2+h^2)^{1/2} + \frac{h^2}{(r^2+h^2)^{\frac{1}{2}}}\right]_0^a \quad \text{(Dwight 203.03)}$$

$$= \frac{\rho_{s0}h}{2\epsilon_0}\left\{\left[(a^2+h^2)^{\frac{1}{2}} + \frac{h^2}{(a^2+h^2)^{\frac{1}{2}}}\right] - (h+h)\right\}$$

$$\vec{E}\big/_{z=h} = \frac{\rho_{s0}h}{2\epsilon_0}\left[\frac{a^2+2h^2-2h(a^2+h^2)^{\frac{1}{2}}}{(a^2+h^2)^{\frac{1}{2}}}\right]\hat{a}_z \quad \text{V/m}$$

[for very large a, i.e., $a \gg h$, $\Rightarrow \vec{E}$ approaches $\frac{\rho_{s0}ha}{2\epsilon_0}\hat{a}_z$, but as $a\to\infty$ so does ρ_s, E_z, so nothing is gained by such an examination, since such a ρ_s is only realistic as an approximation of a distribution for a limited radius]

For the radial componant, we must not forget that the spaial direction of the unit vector $\hat{a}_r$ changes as ϕ changes, so convert $\hat{a}_r$ to a_x, $\hat{a}_y$ components!

$$\int_0^{2\pi}\int_0^a \frac{r^4\hat{a}_r\,dr\,d\varphi}{(r^2+h^2)^{\frac{3}{2}}} = \int_0^{2\pi}\int_0^a \frac{r^4(\cos\varphi\hat{a}_x + \sin\varphi\hat{a}_y)\,dr\,d\varphi}{(r^2+h^2)^{3/2}}$$

$$= \int_0^a \frac{r^4}{(r^2+h^2)^{\frac{3}{2}}}\left[\int_0^{2\pi}(\cos\varphi\hat{a}_x + \sin\varphi\hat{a}_y)\,d\varphi\right]dr = \int_0^a \frac{r^4}{(r^2+h^2)^{\frac{3}{2}}}\left[\sin\varphi\hat{a}_x - \cos\varphi\hat{a}_y\right]_0^{2\pi}dr$$

$$= \int_0^a \frac{r^4}{(r^2+h^2)^{\frac{3}{2}}}\left[0\right]dr\ ! = 0$$

Problem 4.23 ECEN 3613

dielectric sphere, radius a : $\vec{D} = \rho_0 R \hat{a}_R$ C/m²

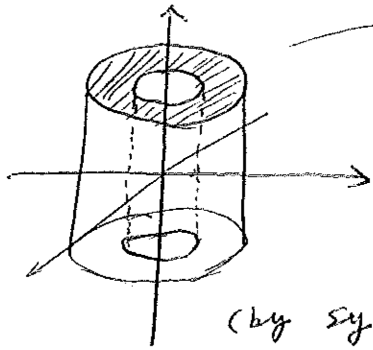
$\Phi_{encl} = \oint_s \vec{D} \cdot d\vec{s}$ $d\vec{s} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_R$, $\theta|_0^{\pi}$, $\phi|_0^{2\pi}$ at $R = a$

$= \int_0^{2\pi} \int_0^{\pi} \rho_0 a^3 \sin\theta \, d\theta \, d\varphi = \rho_0 a^3 \int_0^{2\pi} \left(\int_0^{\pi} \sin\theta \, d\theta \right) d\varphi$

$= \rho_0 a^3 \int_0^{2\pi} 2 \, d\varphi$

$Q_{encl} = \underline{4\pi \rho_0 a^3}$ Coulombs

Problem 4.25 ECEN 3613



— infinitely long shell $1 \le r \le 3$

in which $\rho_v = \rho_{v_0}$ (uniformly distributed)

for $0 \le r < 1m$, $\rho_v = 0$ $\therefore \vec{D} = 0$

for $1 \le r \le 3m$, $\rho_v = \rho_{v_0}$

$\vec{\nabla} \cdot \vec{D} = \rho_{v_0} = \frac{1}{r}\frac{\partial}{\partial r}(r D_r)$

(by symmetry, D_ϕ D_z are zero

so a cylindrical surface at r will be a suitable Gaussian surface

$\vec{\nabla} \cdot \vec{D} = \rho_v$

$\frac{d}{dr}(r D_r) = \rho_{v_0} r$

$d(r D_r) = \rho_{v_0} r\, dr$

$r D_r = \frac{\rho_{v_0} r^2}{2} + k$

$D_r = \frac{\rho_{v_0} r}{2} + \frac{k}{r}$

at $r=1$, $D_r = 0$ $\therefore k = -\frac{\rho_{v_0}}{2}$

$D_r = \frac{\rho_{v_0}}{2}(r - \frac{1}{r})$

$\vec{D} = \frac{\rho_{v_0}(r^2-1)}{2r}\hat{a}_r$

or $\lim_{h\to\infty}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi} D_r\, r\, d\phi\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi}\int_1^r \rho_{v_0}\, r\, dr\, d\phi\, dz\right]$

$2\pi r h D_r = \frac{2\pi \rho_{v_0} h (r^2-1)}{2}$

$D_r = \frac{\rho_{v_0}(r^2-1)}{2r}$

$\vec{D} = \frac{\rho_{v_0}(r^2-1)}{2r}\hat{a}_r$

(same result)

For $3 < r$, all of the ρ_{v_0} is enclosed, so at $r=3$, $D_r = \frac{4}{3}\rho_{v_0}$

$\vec{\nabla} \cdot \vec{D} = 0$

$\frac{d(r D_r)}{dr} = 0$

$d(r D_r) = 0$

$r D_r = k$

at $r=3$, $k = 3 D_{r_3} = 3(\frac{4}{3}\rho_{v_0})$

$k = 4\rho_{v_0}$

$D_r = 4\rho_{v_0}/r$

$\vec{D} = \frac{4\rho_{v_0}}{r}\hat{a}_r$

or $\lim_{h\to\infty}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi} D_r\, r\, d\phi\, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi}\int_1^3 \rho_{v_0}\, r\, dr\, d\phi\, dz\right]$

$\lim_{h\to\infty}\left[2\pi r h D_r = 2\pi \rho_{v_0} h \left(\frac{r^2}{2}\right)\Big|_1^3\right]$

$r D_r = \rho_{v_0}(\frac{9}{2} - \frac{1}{2})$

$D_r = \frac{4\rho_{v_0}}{r}$

$\vec{D} = \frac{4\rho_{v_0}}{r}\hat{a}_r$

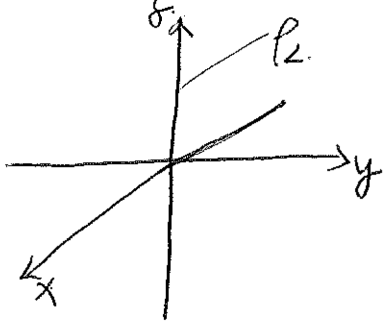
(same result)

$0 \le r < 1$ $\vec{D} = 0$

$1 \le r \le 3$ $\vec{D} = \frac{\rho_{v_0}(r^2-1)}{2r}\hat{a}_r$

$3 < r$ $\vec{D} = \frac{4\rho_{v_0}}{r}\hat{a}_r$

Problem 4.30 ECEN 3613



using a Gaussian surface $ds = r\,d\phi\,dz$ w/ D_r only

$$Q = \int_l \rho_L\,dl = \oint_S \vec{D}\cdot d\vec{S} = \underbrace{\int_{top}}_{0} + \underbrace{\int_{bottom}}_{0} + \lim_{h\to\infty}\left(\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi} r D_r\,d\phi\,dz\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_L\,dz$$

(only a D_r component)

$$= \lim_{h\to\infty}\left[2\pi r D_r h = \rho_L h\right]$$

$$\Rightarrow D_r = \rho_L / 2\pi r$$

method 1 $\Rightarrow E_r = \frac{D_r}{\epsilon} = \rho_L / 2\pi\epsilon r$, $\quad \vec{E} = \frac{\rho_L}{2\pi\epsilon r}\hat{a}_r$

$$V_{12} = -\int_{P_2}^{P_1} \vec{E}\cdot d\vec{l} = \frac{-\rho_L}{2\pi\epsilon}\int_{r_2}^{r_1}\frac{dr}{r} = \frac{\rho_L}{2\pi\epsilon}\ln(r)\Big|_{r_1}^{r_2} = \frac{\rho_L}{2\pi\epsilon}\ln\left(\frac{r_2}{r_1}\right)$$

$$\boxed{V_{12} = \frac{\rho_L}{2\pi\epsilon}\ln\left(\frac{r_2}{r_1}\right)}$$

method 2 $\vec{E} = -\vec{\nabla} V$

$\therefore E_r = \frac{-\partial V}{\partial r} = -\frac{dV}{dr}$ (no E_ϕ or E_z components)

$$\frac{dV}{dr} = -\rho_L / 2\pi\epsilon r$$

$$\int_2^1 dV = -\frac{\rho_L}{2\pi\epsilon}\int_{r_2}^{r_1}\frac{dr}{r}$$

$$V_1 - V_2 = V_{12} = \rho_L / 2\pi\epsilon \ln(r)\Big|_{r_1}^{r_2} = \frac{\rho_L}{2\pi\epsilon}\left[\ln(r_2) - \ln(r_1)\right]$$

$$V_{12} = \frac{\rho_L}{2\pi\epsilon}\ln\left(\frac{r_2}{r_1}\right)$$

Problem 4.37 ECEN 3613

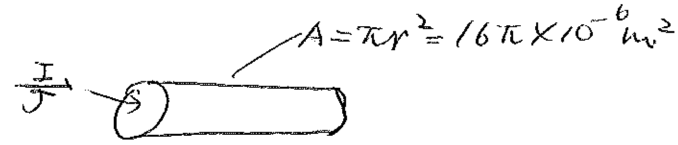
Cylindrical bar, So: $r = 4mm = 4\times10^{-3}m$, $l = 8cm = 8\times10^{-2}m$

$V = 5$ volts $\mu_e = 0.13 m^2/Vs$ $N_e = 1.5\times10^{16}$ electrons/m3

$\mu_h = 0.05 m^2/Vs$ $N_h = N_e$

(a) $\sigma = (N_e\mu_e + N_h\mu_h)e = (1.5\times10^{16})(0.13+0.05)(1.6\times10^{-19}) = 0.432\times10^{-3}$

$\sigma = \underline{0.432m}$ Siemens/meter



(b) $I = \int_S \vec{J}\cdot d\vec{s}$ $\vec{J} = \sigma\vec{E}$

$V_{+-} = -\int_-^+ \vec{E}\cdot d\vec{l} \Rightarrow V_{+-} = E\,l$ $E = \frac{V}{l} = \frac{5}{8\times10^{-2}} = 62.5 V/m$

$I = \sigma(62.5)16\pi\times10^{-6} = (0.432\times10^{-3})(62.5)(16\pi\times10^{-6}) = 1.357\times10^{-6}$

$I = \underline{1.357\mu}$ Amps

(c) $\vec{u}_e = -\mu_e\vec{E} \Rightarrow u_e = -(0.13)(62.5) = -8.13 m/sec \Rightarrow \vec{u}_e = -8.13\hat{a}_E m/s$

$\vec{u}_h = \mu_h\vec{E} \Rightarrow u_h = (0.05)(62.5) = 3.13 m/sec \Rightarrow \vec{u}_h = 3.13\hat{a}_E m/s$

(d) $R = \frac{l}{\sigma A} = \frac{8\times10^{-2}}{(0.432\times10^{-3})(16\pi\times10^{-6})} = 0.368\times10^{7}$

$\underline{R = 3.68M\Omega}$

(e) $P = \int_v \sigma E^2 dv = \sigma E^2 l A = (0.432\times10^{-3})(62.5)^2(8\times10^{-2})(16\pi\times10^{-6})$

$= 67.85\times10^{-7}$

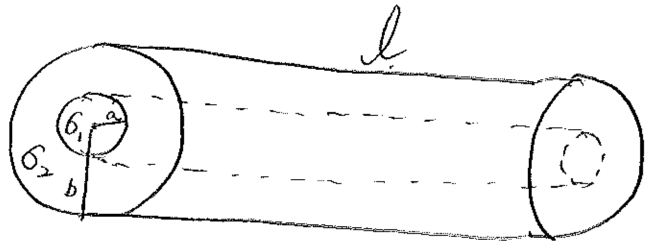
$P = \underline{6.785\ \mu\ Watts}$

$\left(= RI^2 = VI = \frac{V^2}{R}\right.$

$= (5)(1.357\times10^{-6}) = 6.785\times10^{-6}$

$\left.= 6.785\ \mu Watts\right)$

Problem 4.60 ECEN 3613



$A_1 = \pi a^2$

$A_2 = \pi b^2 - \pi a^2 = \pi(b^2 - a^2)$

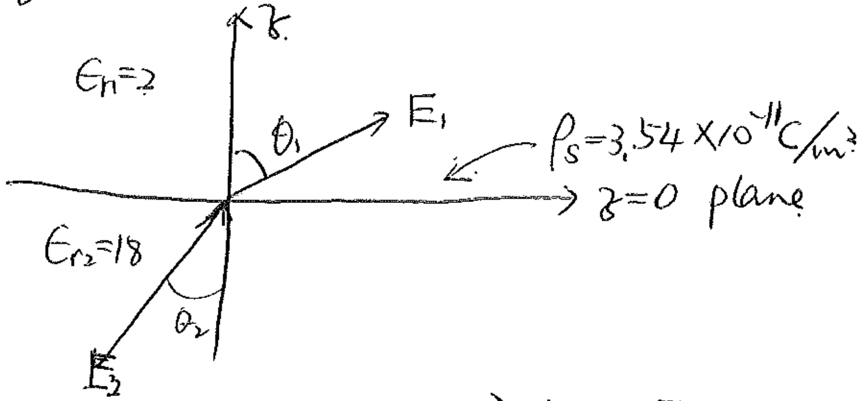
$$R_1 = \frac{l}{\sigma_1 A_1} = \frac{l}{\sigma_1 \pi a^2}$$

$$R_2 = \frac{l}{\sigma_2 \pi (b^2 - a^2)}$$

$$R = R_1 \parallel R_2 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{1}{\frac{\sigma_1 \pi a^2}{l} + \frac{\sigma_2 \pi (b^2 - a^2)}{l}} = \frac{l}{\pi[\sigma_1 a^2 + \sigma_2 (b^2 - a^2)]}$$

Problem 4.43 ECEN 3613

(using Fig 4.19, P179)



$\vec{E}_2 = 3\hat{a}_x - 2\hat{a}_y + 2\hat{a}_z$ V/m

$E_{1t} = E_{2t} \Rightarrow E_{1x} = E_{2x} \quad E_{1y} = E_{2y}$

$E_{1x} = 3 \quad E_{1y} = -2$

$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s \Rightarrow \epsilon_{r1} E_{1n} = \epsilon_{r2} E_{2n} + \rho_s/\epsilon_0 \Rightarrow E_{1z} = \frac{\epsilon_{r2}}{\epsilon_{r1}} E_{2z} + \frac{\rho_s}{\epsilon_{r1}\epsilon_0}$

$$E_{1z} = \left(\frac{18}{2}\right)2 + \frac{3.54\times10^{-11}}{2(8.854\times10^{-12})}$$

$$E_{1z} = 18 + 2 = 20$$

$\Rightarrow \vec{E}_1 = 3\hat{a}_x - 2\hat{a}_y + 20\hat{a}_z$ Volts/meter

$$\sin\theta_2 = \frac{E_{2t}}{E_2} = \frac{\sqrt{3^2+2^2}}{\sqrt{3^2+2^2+2^2}} = \frac{\sqrt{13}}{\sqrt{17}} = 0.8745$$

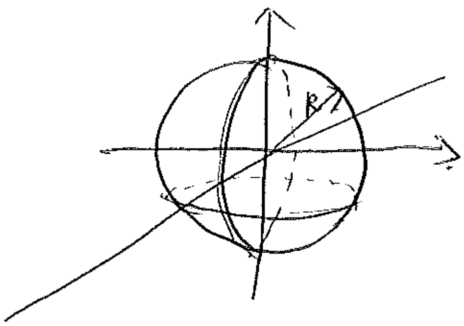
$$\theta_2 = \sin^{-1}(0.8745)$$

$$\underline{\underline{\theta_2 = 60.98^\circ}}$$

Problem 4.46 ECEN 3613

Conducting sphere, centered at the origin, with radius of 5 cm ($R = 5\times10^{-2}$ m)

$\vec{E} = 150\,\hat{a}_R$ V/m



At the surface, $E_t = 0$ & $E_n = 150$

$\epsilon_0 E_n = \rho_s$

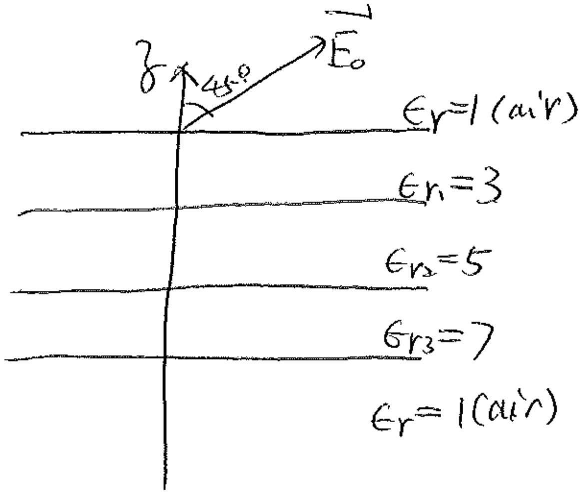
$$Q = \int_S \rho_s\, ds = \int_0^{2\pi}\int_0^{\pi} \rho_s R^2 \sin\theta\, d\theta\, d\phi$$

$$Q = 150\,\epsilon_0\,(5\times10^{-2})^2\,4\pi$$

$$Q = 150\,(25\times10^{-4})\,\frac{4\pi}{36\pi}\times10^{-9} = 416.7\times10^{-13}$$

$Q = 41.67$ p. Coulombs

Problem 4.47 ECEN 3613



with $\rho_s = 0$ at the boundaries.

$D_{n1} = D_{n2} \Rightarrow \epsilon_1 E_{n1} = \epsilon_2 E_{n2}$

$E_{t1} = E_{t2}$

$E_{n2} = \frac{\epsilon_1}{\epsilon_2} E_{n1}$

$E_n = |E| \cos\theta$

$E_t = |E| \sin\theta$

$E_t / E_n = \frac{|E| \sin\theta}{|E| \cos\theta} = \tan\theta$

$\frac{E_{t1}}{E_{n1}} = \tan\theta_1 \qquad \frac{E_{t2}}{E_{n2}} = \tan\theta_2$

at the 1-2 boundry $\Rightarrow \frac{E_{t1}}{\frac{\epsilon_1}{\epsilon_2} E_{n1}} = \tan\theta_2 \Longrightarrow \frac{E_{t1}}{E_{n1}} = \frac{\epsilon_{r1}}{\epsilon_{r2}} \tan\theta_2$

$\tan\theta_1 = \frac{\epsilon_{r1}}{\epsilon_{r2}} \tan\theta_2$

or $\tan\theta_2 = \frac{\epsilon_{r2}}{\epsilon_{r1}} \tan\theta_1$ (same as Eq 4.88, P166)

∴ for the air − 1 boundry, $\tan\theta_1 = \frac{3}{1} \tan 45° = 3 \Rightarrow \theta_{a1} = 71.57°$

1 − 2 ″ , $\tan\theta_2 = \frac{5}{3} \tan 71.57 = \frac{5}{3}(3) \Rightarrow \theta_{12} = 78.69°$

2 − 3 ″ , $\tan\theta_3 = \frac{7}{5} \tan 78.67 = \frac{7}{5}(5) \Rightarrow \theta_{23} = 81.87°$

3 − a ″ , $\tan\theta_a = \frac{1}{7} \tan = \frac{1}{7}(7) \Rightarrow \underline{\theta_{3a} = 45°}$

Problem 4.48 ECEN 3613

using Eqn 4.131, p191, $F = \frac{\epsilon A E^2}{2}$

$A = 5\,cm^2 = 5\times10^{-4}\,m^2$

$d = 2\,cm = 2\times10^{-2}\,m$

$\epsilon_r = 4$

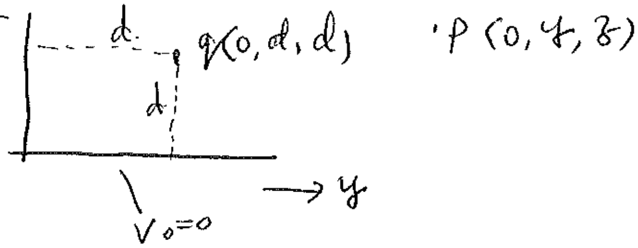
$V = 50\,V$

$E = \frac{V}{d} \Rightarrow F = \frac{\epsilon_r \epsilon_0 A}{2}\frac{V^2}{d^2} = \epsilon_r \epsilon_0 A V^2 / 2d^2$

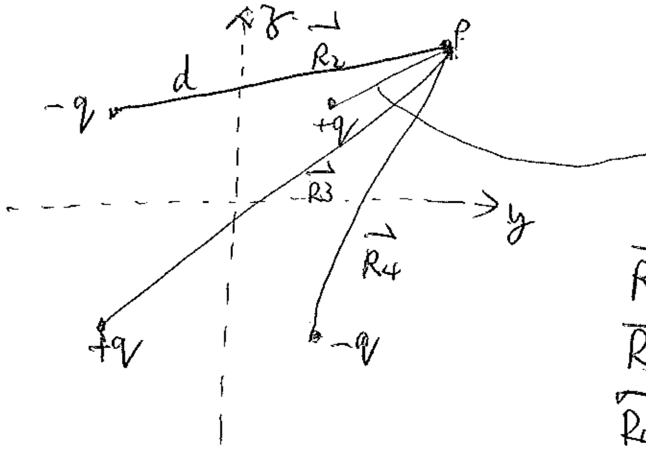
$$F = \frac{4(8.854\times10^{-12})(5\times10^{-4})(2500)}{2(4\times10^{-4})} = 55338\times10^{-12} = 55.338\times10^{-9}$$

$F =$ 55.34 n Newtons (attraction)

Problem 4.56 ECEN 3613



By images (like "looking" in a pair of mirrors at right angles)



$$\vec{R}_1 = (y-d)\hat{a}_y + (z-d)\hat{a}_z$$

(no x component in the x=0 plane)

$$\vec{R}_2 = (y+d)\hat{a}_y + (z-d)\hat{a}_z$$

$$\vec{R}_3 = (y+d)\hat{a}_y + (z+d)\hat{a}_z$$

$$\vec{R}_4 = (y-d)\hat{a}_y + (z+d)\hat{a}_z$$

$$V\Big|_{x=0} = \sum_{i=1}^{4} V_i = \frac{q}{4\pi\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$$

$$V\Big|_{x=0} = \frac{q}{4\pi\epsilon}\left[\frac{1}{\sqrt{(y-d)^2+(z-d)^2}} - \frac{1}{\sqrt{(y+d)^2+(z-d)^2}} + \frac{1}{\sqrt{(y+d)^2+(z+d)^2}} - \frac{1}{\sqrt{(y-d)^2+(z+d)^2}}\right]$$

$$\vec{E}\Big|_{x=0} = -\vec{\nabla} V \text{ or } \vec{E} = \sum_{i=1}^{4} \vec{E}_i = \frac{q}{4\pi\epsilon}\left[\frac{\vec{R}_1}{R_1^3} - \frac{\vec{R}_2}{R_2^3} + \frac{\vec{R}_3}{R_3^3} - \frac{\vec{R}_4}{R_4^3}\right]$$

$$\vec{E}\Big|_{x=0} = \frac{q}{4\pi\epsilon}\left[\frac{(y-d)\hat{a}_y + (z-d)\hat{a}_z}{[(y-d)^2+(z-d)^2]^{\frac{3}{2}}} - \frac{(y+d)\hat{a}_y + (z-d)\hat{a}_z}{[(y+d)^2+(z-d)^2]^{\frac{3}{2}}} + \frac{(y+d)\hat{a}_y + (z+d)\hat{a}_z}{[(y+d)^2+(z+d)^2]^{\frac{3}{2}}} - \frac{(y-d)\hat{a}_y + (z+d)\hat{a}_z}{[(y-d)^2+(z+d)^2]^{\frac{3}{2}}}\right]$$

[Note, to add the dependence on x, each $\vec{R}$ & R must be modified by adding $x\hat{a}_x$ to each $\vec{R}$ term, and x^2 to each R^2 set i.e., $\vec{R}_1$ becomes $x\hat{a}_x + (y+d)\hat{a}_y + (z-d)\hat{a}_z$

and R_1 " $\sqrt{x^2+(y+d)^2+(z-d)^2}$, etc.

this is eacily handled by a computer!]